Numerical Solution of the Nonlocal Singularly Perturbed Problem

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Abstract: *The study is concerned with a different perspective which the numerical solution of the singularly perturbed nonlinear boundary value problem with integral boundary condition using finite difference method on Bakhvalov mesh. So, we show some properties of the exact solution. We establish uniformly convergent finite difference scheme on Bakhvalov mesh. The error analysis for the difference scheme is performed. The numerical experiment implies that the method is the first order convergent in the discrete maximum norm, independently of ε- singular perturbation parameter with effective and efficient iterative algorithm. The numerical results are shown in table and graphs.*

Keywords: *Singular perturbation, finite difference method, Bakhvalov mesh, uniformly convergence, integral condition.*

1. Introduction

This paper focused on the first-order nonlinear singularly perturbed differential equation.

$$\varepsilon u' + f(x,u) = 0, \qquad x \in (0,L], \qquad L > 0 \tag{1}$$

subject to the integral boundary condition

$$u(0) = \beta u(L) + \int_0^L b(s)u(s)ds + d \tag{2}$$

where ε is a small perturbation parameter $(0 < \varepsilon \ll 1)$. $\alpha,\ \beta,\ d$ are constant, which are independent of ε. We assume that $b(x)$, $f(x,u)$ are sufficiently smooth functions in the intervals $(0,L]$ and $(0,L]xR$. Also, $\frac{\partial f}{\partial u} \geq \alpha > 0$. The solution of (1) has in general a boundary layer with O(ε) near to $x = 0$. The first time, nonlocal problems were studied by Bitsadze and Samarskii [16]. Differential equations with conditions which connect the values of the unknown solution at the boundary with values in the interior are known as nonlocal boundary value problems [16]. Existence and uniqueness of nonlocal problems can be seen in [1-3]. Singular perturbation problems can also arise in fluid mechanics, quantum mechanics, hydromechanical problems, chemical-reactor theory, control theory, oceanography, meteorology, electrical networks and other physical models. The subject of nonlocal boundary value problems for singularly perturbed differential equations has been studied by many authors. For example, Cziegis [4] studied the numerical solution of singularly perturbed nonlocal problem. Cziegis [5] analyzed the difference schemes for problems with nonlocal conditions. Amiraliyev and Çakır [8] applied the difference method on a Shishkin mesh to the Singularly perturbed three-point boundary value problem. Amiraliyev and Çakır [9] studied numerical solution of the singularly perturbed problem with nonlocal boundary condition. Çakır and Arslan [14] analyzed a numerical method for nonlinear singularly perturbed multi-point boundary value problem. Adzic and Ovcin [11] studied nonlinear spp with nonlocal boundary conditions and spectral approximation. Amiraliyev, Amiraliyeva and Kudu [12] applied a numerical treatment for singularly perturbed differential equations with integral boundary condition. Herceg [6] studied the numerical solution of a singularly perturbed nonlocal problem. Herceg [7] researched solving a nonlocal singularly perturbed problem by splines in tension. Çakır [10] studied uniform second–order difference method for a singularly perturbed three-point boundary value problem. Geng [15] applied a numerical algorithm for nonlinear multi-point boundary value problems. Çakır [13] obtained a numerical study on the difference solution of singularly perturbed semilinear problem with integral boundary condition. It is well known that standard discretization methods for solving nonlocal singular perturbation problem are unstable and these don't give accurate results for ε. Therefore, it is very important to find suitable numerical methods to these problem. So, we use finite difference method in this paper. According to this method, we will analyze that this method for the numerical solution of the nonlocal problem (1)-(2) is uniformly convergent of first order on Bakhvalov mesh, in discrete maximum norm, indepedently of singular perturbation parameter ε. In section 2, some properties of the exact solution of problem described in (1)-(2) is investigated. According to the perturbation parameter, by the method of integral identities with the use exponential basisfunctions and interpolating quadrature rules with the weight and remainder terms in integral form uniformly convergent finite difference scheme on Bakhvalov mesh is established in section 3. The error analysis for the difference scheme is performed in section 4. In section5, numerical examples are presented to find the solution of approximation. In

this paper, we use $\|g(x)\|_\infty$ for the continuous maximum norm on the corresponding interval. $g(x)$ is any continuous function.

2. Properties of the Exact Solution

The exact solution of (1)-(2) has some properties, which are needed in later sections for analysis of the numerical solution. Thus, we describe these properties with Lemma 2.1.

Lemma 2.1. If $\frac{\partial f(x,u)}{\partial u} \geq \alpha > 0$ and $f(x,u) \in (0,L]xR$, the estimates

$$\| u(x) \| \leq C_0 \tag{3}$$

where

$$C_0 = |u(0)|e^{\frac{-\alpha t}{\varepsilon}} + \left(1 - e^{\frac{-\alpha T}{\varepsilon}}\right)\alpha^{-1}\|F\|_\infty, \qquad F(x) = -f(x,0)$$

and

$$\|u'(x)\| \leq C\left\{1 + \frac{1}{\varepsilon}e^{\frac{-\alpha x}{\varepsilon}}\right\} \tag{4}$$

hold for the solution $u(x)$, where C_0 and C are constant independent of ε and h.

Proof. We use intermediate value theorem for $f(x,u)$ in Equation (1)

$$\frac{f(x,u) - f(x,0)}{u} = \frac{\partial f(x,\tilde{u})}{\partial u}, \qquad \tilde{u} = \gamma u, \qquad 0 < \gamma < 1$$

and

$$a(x) = \frac{\partial f(x,\tilde{u})}{\partial u}.$$

Thus, we obtain the following linear equation,

$$\varepsilon u' + a(x)u(x) = F(x) \tag{5}$$

We can write the solution of the Equation (5) as follows:

$$u(x) = u(0)e^{\frac{-1}{\varepsilon}\int_0^x a(\tau)d\tau} + \frac{1}{\varepsilon}\int_0^x F(\xi)e^{\frac{-1}{\varepsilon}\int_\xi^x a(\tau)d\tau}\,d\xi \tag{6}$$

In Equation (5), $x = T$ and $x = s$, we obtain Eq (7)-Eq(9)

$$u(T) = u(0)e^{\frac{-1}{\varepsilon}\int_0^T a(\tau)d\tau} + \frac{1}{\varepsilon}\int_0^T F(\xi)e^{\frac{-1}{\varepsilon}\int_\xi^T a(\tau)d\tau}\,d\xi \tag{7}$$

$$u(s) = u(0)e^{\frac{-1}{\varepsilon}\int_0^s a(\zeta)d\zeta} + \frac{1}{\varepsilon}\int_0^s F(\xi)e^{\frac{-1}{\varepsilon}\int_\xi^s a(\zeta)d\zeta}\,d\xi \tag{8}$$

and

$$u(0) = \frac{d + \frac{\beta}{\varepsilon}\int_0^T F(\xi)e^{\frac{-1}{\varepsilon}\int_\xi^T a(\tau)d\tau}\,d\xi + \frac{1}{\varepsilon}\int_0^T b(s)\,[\int_0^T F(\xi)e^{\frac{-1}{\varepsilon}\int_\xi^T a(\tau)d\tau}d\xi]ds}{1 - \beta e^{\frac{-1}{\varepsilon}\int_0^T a(\tau)d\tau} - \int_0^T b(s)e^{\frac{-1}{\varepsilon}\int_0^s a(\tau)d\tau}\,ds}. \tag{9}$$

Now, we evaluate to Equation (9) for $a(\tau) \geq \alpha > 0$ and $b^* = max|b(x)|$.

First, we prove the estimate the following inequality,

$$1 - \beta e^{\frac{-1}{\varepsilon}\int_0^T a(\tau)d\tau} - \int_0^T b(s)e^{\frac{-1}{\varepsilon}\int_0^s a(\tau)d\tau}\,ds \geq 1 - \beta e^{\frac{-\alpha T}{\varepsilon}} - b^*\int_0^T e^{\frac{-\alpha s}{\varepsilon}}ds \geq 1 - \beta e^{\frac{-\alpha T}{\varepsilon}} - b^*\left(1 - e^{\frac{-\alpha T}{\varepsilon}}\right) \geq C_0$$
$$> 0$$

Second, we prove the estimate the following inequality,

$$d+\frac{\beta}{\varepsilon}\int_0^T F(\xi)e^{\frac{-1}{\varepsilon}\int_\xi^T a(\tau)d\tau}\,d\xi+\frac{1}{\varepsilon}\int_0^T b(s)\,[\int_0^T F(\xi)e^{\frac{-1}{\varepsilon}\int_\xi^T a(\tau)d\tau}\,d\xi]ds$$

$$\leq |d|+|\beta|\frac{1}{\varepsilon}\left[\|F\|_\infty\int_0^T e^{\frac{-1}{\varepsilon}\int_\xi^T a(\tau)d\tau}\,d\boldsymbol{\xi}\right]+\frac{1}{\varepsilon}\int_0^T b(s)\left[\int_0^T F(\xi)e^{\frac{-1}{\varepsilon}\int_\xi^T a(\tau)d\tau}\,d\xi\right]ds$$

$$=|d|+|\beta|\frac{1}{\varepsilon}\left[\|F\|_\infty\int_0^T e^{\frac{-\alpha}{\varepsilon}(T-\xi)}\,d\boldsymbol{\xi}\right]+\frac{1}{\alpha}\|F\|_\infty\int_0^T b(s)\left[1-e^{\frac{-\alpha s}{\varepsilon}}\right]ds$$

$$\leq |d|+|\beta|\frac{1}{\varepsilon}\left[\|F\|_\infty\frac{\varepsilon}{\alpha}\left(1-e^{\frac{-\alpha T}{\varepsilon}}\right)\right]+\frac{1}{\alpha}\|F\|_\infty\int_0^T b(s)\left[1-e^{\frac{-\alpha s}{\varepsilon}}\right]ds$$

$$=|d|+|\beta|\left[\|F\|_\infty\frac{1}{\alpha}\left(1-e^{\frac{-\alpha T}{\varepsilon}}\right)\right]+\frac{1}{\alpha}\|F\|_\infty\int_0^T b(s)\left[1-e^{\frac{-\alpha s}{\varepsilon}}\right]ds$$

$$\leq |d|+|\beta|\|F\|_\infty\frac{1}{\alpha}+\frac{1}{\alpha}\|F\|_\infty\|b\|_1\leq C_1$$

where

$$\int_0^T b(s)\,ds=\|b\|_1.$$

Consequently, we obtain the following inequality:

$$u(0)\leq C$$

and

$$|u(x)|\leq |u(0)|e^{\frac{-\alpha x}{\varepsilon}}+\left(1-e^{\frac{-\alpha T}{\varepsilon}}\right)\alpha^{-1}\|F\|_\infty$$

$$\leq \left[{C_0}^{-1}|d|+{C_0}^{-1}(|\beta|+\|b\|_1)\alpha^{-1}\|F\|_\infty\right]\left[e^{\frac{-\alpha x}{\varepsilon}}+\left(1-e^{\frac{-\alpha T}{\varepsilon}}\right)\alpha^{-1}\|F\|_\infty\right]$$

$$\leq [{C_0}^{-1}|d|+{C_0}^{-1}(|\beta|+\|b\|_1)\alpha^{-1}\|F\|_\infty\leq\ C_0.$$

We now prove the estimate the Equation (4) :

$$\left|\frac{\partial f}{\partial x}\right|\leq C,\ \ |u(x)|\leq C_0.$$

If the above inequalities replace the Equation (6), we have the following inequality

$$|u(x)|\leq |u(0)|e^{\frac{-\alpha x}{\varepsilon}}+\left(1-e^{\frac{-\alpha T}{\varepsilon}}\right)\alpha^{-1}\|F\|_\infty \tag{10}$$

where $|u(0)|\leq C,\quad \alpha^{-1}\|F\|_\infty\leq C.$

If we take derivative of the Equation (10) we obtain as following:

$$|u'(x)|\leq |u(0)|\frac{C\alpha}{\varepsilon}e^{\frac{-\alpha x}{\varepsilon}}+\frac{\alpha}{\varepsilon}e^{\frac{-\alpha x}{\varepsilon}}\leq C\left(1+\frac{1}{\varepsilon}e^{\frac{-\alpha x}{\varepsilon}}\right)$$

thus, the proving of the Lemma 1 is completed.

3. The Establishment of Difference Scheme

Let us consider the following any non-uniform mesh on $[0,L]$,

$$\omega_N=\{x_0<x_1<\cdots<x_{N-1}<x_N\ ,\qquad h_i=x_i-x_{i-1}\}$$

$$\overline{\omega}_N=\omega_N\cup\{x=0,x=L\}.$$

First, we will construct the difference scheme for the Equation (1). First, we integrate the Equation (1) over (x_{i-1},x_i)

$$h_i^{-1}\int_{x_{i-1}}^{x_i} \varepsilon u'(x))dx + h_i^{-1}\int_{x_{i-1}}^{x_i} f(x,u)dx = 0, \qquad i = \overline{1,N-1} \tag{11}$$

and we obtain the following inequality

$$h_i^{-1}\varepsilon\left[u(x_i) - u(x_{i-1})\right) + h_i^{-1}\int_{x_{i-1}}^{x_i} f(x,u)dx = \varepsilon u_{\bar{x},i} + h_i^{-1}\int_{x_{i-1}}^{x_i} f(x,u)dx = 0$$

where

$$R_i = -h_i^{-1}\int_{x_{i-1}}^{x_i} (x_i - x_{i-1})f'(x,u)dx. \tag{12}$$

So, from the Equation (12) the difference scheme is defined by

$$\varepsilon u_{\bar{x},i} + f(x_i,u_i) + R_i = 0, \qquad i = \overline{1,N} \tag{13}$$

Second, we define an approximation for the boundary condition of the Equation (1).

$$u_0 = \beta u_N + \sum_{i=1}^{N}\int_{x_{i-1}}^{x_i} b(s)u(s)ds + d \tag{14}$$

where

$$\sum_{i=1}^{N}\int_{x_{i-1}}^{x_i} b(s)u(s)ds = \sum_{i=1}^{N}\int_{x_{i-1}}^{x_i} [b(s)u(s) - b_i u_i]ds$$
$$+\sum_{i=1}^{N} b_i u_i h_i = r + \sum_{i=1}^{N} b_i u_i h_i \tag{15}$$

and remainder term

$$r = \sum_{i=1}^{N}\int_{x_{i-1}}^{x_i} (x - x_{i-1})\frac{d}{dt}[b(t)u(t)]d\boldsymbol{t}\,. \tag{16}$$

$$u_0 = \beta u_N + \sum_{i=1}^{N} h_i b_i u_i + d + r. \tag{17}$$

Thus, by neglecting R_i and r in the Equation (13) and the Equation (17), we suggest the following difference scheme for approximating the Equations (1)-(2) :

$$\varepsilon y_{\bar{x},i} + f(x_i,y_i) = 0\,, i = \overline{1,N} \tag{18}$$

$$y_0 = \beta y_N + \sum_{i=1}^{N} h_i b_i y_i + d \tag{19}$$

4. Bakhvalov Mesh

We use Bakhvalov mesh, because the difference scheme (18)-(19) must be ε −uniform convergent. According to the transition point following as:

$$\sigma = min\left\{\frac{L}{2}, -\alpha^{-1}\varepsilon ln\varepsilon\right\}$$

the piecewise uniform mesh divides each of the interval $[0,\sigma]$ and $[\sigma,L]$ into $\frac{N}{2}$ equidistant subintervals. The mesh point x_i is defined as follows:

$\sigma < \frac{L}{2}\ ise,\ x_i \in [0,\sigma],\ x_i = -\alpha^{-1}\varepsilon\ln\left[1 - (1-\varepsilon)\frac{2i}{N}\right],\ i = 0,1,\dots,\frac{N}{2}$
$\sigma = \frac{L}{2}\ ise\ ,\ \ x_i \in [0,\sigma], x_i = -\alpha^{-1}\varepsilon\ln\left[1 - \left(1 - e^{\frac{-\alpha L}{2\varepsilon}}\right)\frac{2i}{N}\right], i = 0,1,\dots,\frac{N}{2}$
$x_i \in [\sigma,L],\ x_i = \sigma + \left(i - \frac{N}{2}\right)h,\ i = \frac{N}{2} + 1,\dots,N,\ h = \frac{2(L-\sigma)}{N}.$

5. Uniform Error Estimates

In this section, we obtain the convergence of the method. First, we give the error function $z_i = y_i - u_i,\ i = \overline{0,N},\ z_i$ is the solution of the discrete problem,

$$\varepsilon z_{\bar{x},i} + f(x_i, y_i) - f(x_i, u_i) = R_i, \qquad i = \overline{1,N} \tag{20}$$

$$z_0 = \beta z_N + \sum_{i=1}^{N} h_i b_i z_i - r \tag{21}$$

where R_i and r are defined by Equation (12) and Equation (16).

Lemma 5.1. Let z_i be the solution to Equation (20),

$$|b_*| = \max_i |b_i|, \quad \rho_k = \frac{h_k}{\varepsilon}, \quad k = 1,2 \text{ and } \alpha = \max_i |a_i|$$

and

$$1 - \beta \left| \left(\frac{1}{1 + \alpha\rho_i} \right)^N \right| - b_* \sum_{i=1}^{N} h_i \left| \left(\frac{1}{1 + \alpha\rho_k} \right)^i \right| \neq 0.$$

Then the estimate

$$\|z\|_{\infty,\bar{\omega}_N} \leq C\left(\| R\|_{\infty,\bar{\omega}_N} + |r|\right) \tag{22}$$

holds.

Proof. We use intermediate value theorem to rewrite the Equation (20) as follows:

$$f(x_i, y_i) - f(x_i, u_i) = \tilde{f}_u (y_i - u_i) = \tilde{f}_u z_i$$

Here we assumed that

$$\tilde{f}_u = a_i.$$

Thus, we can write

$$\varepsilon z_{\bar{x},i} + a_i z_i = R_i, \qquad i = \overline{1, N-1} \tag{23}$$

$$z_0 = \beta z_N + \sum_{i=1}^{N} h_i b_i z_i - r. \tag{24}$$

Rearranging Equation (23) gives

$$z_i = \frac{\varepsilon}{\varepsilon + a_i h_i} z_{i-1} + \frac{h_i R_i}{\varepsilon + a_i h_i}$$

and

$$z_i = z_0 [\prod_{k=1}^{i} q_k] + \sum_{k=1}^{i} \varphi_k [\prod_{j=k+1}^{i} q_j]$$

where

$$q_k = \frac{\varepsilon}{\varepsilon + a_k h_k}$$

and

$$\varphi_k = \frac{h_k R_k}{\varepsilon + a_k h_k}.$$

According to the maximum principle for Equations (23)-(24), we have the following inequality

$$\|z\|_{\infty,\bar{\omega}_N} \leq |z_0| + \alpha^{-1} \|R\|_{\infty,\bar{\omega}_N}. \tag{25}$$

Now, we evaluate the Equation (24). After a simple calculation we obtain

$$|z_0| \le \beta |z_N| + b_* \sum_{i=1}^{N} h_i |z_i| + |r| \tag{26}$$

where

$$|b_*| = \max_i |b_i|$$

and

$$z_N = z_0 \emptyset_N + \sum_{k=1}^{N} \varphi_k \emptyset_{N-k}. \tag{27}$$

The Equation (27) write in the Equation (26), we have

$$|z_0| \le \beta \left| z_0 \emptyset_N + \sum_{k=1}^{N} \varphi_k \emptyset_{N-k} \right| + b_* \sum_{i=1}^{N} h_i \left| z_0 \emptyset_i + \sum_{k=1}^{i} \varphi_k \emptyset_{i-k} \right| + |r|$$

After doing some arragement,

$$|z_0| \le \frac{\beta \left|\sum_{k=1}^{N} \varphi_k \emptyset_{N-k}\right| + b_* \sum_{i=1}^{N} h_i \left|\sum_{k=1}^{i} \varphi_k \emptyset_{i-k}\right| + |r|}{(1 - \beta |\,\emptyset_N| - b_* \sum_{i=1}^{N} h_i |\emptyset_i|)} \tag{28}$$

it follows that

$$|z_0| \le \frac{\beta \left|\sum_{k=1}^{N} \left(\frac{\rho_k R_k}{1+\alpha\rho_k}\right)\left(\frac{1}{1+\alpha\rho_k}\right)^N\right| + b_* \sum_{i=1}^{N} h_i \left|\sum_{k=1}^{i} \left(\frac{\rho_k R_k}{1+\alpha\rho_k}\right)\left(\frac{1}{1+\alpha\rho_k}\right)^i\right| + |r|}{\left(1 - \beta \left|\left(\frac{1}{1+\alpha\rho_i}\right)^N\right| - b_* \sum_{i=1}^{N} h_i \left|\left(\frac{1}{1+\alpha\rho_k}\right)^i\right|\right)}$$
$$\le C(\|R\|_{\infty,\bar{\omega}_N} + |r|\,) \tag{29}$$

where

$$\rho_k = \frac{h_k}{\varepsilon},\ k = 1{,}2 \ \text{ and } \ \alpha = \max_i |a_i|.$$

If the Equation(29) write in the Equation (25), we have

$$\|z\|_{\infty,\bar{\omega}_N} \le C\big(\|R\|_{\infty,\bar{\omega}_N} + |r|\big) + \alpha^{-1}\|R\|_{\infty,\bar{\omega}_N}$$
$$\le C\big(\|R\|_{\infty,\bar{\omega}_N} + |r|\big).$$

Lemma 5. 2. Under the assumptions of Lemma 1 and $\frac{\partial f}{\partial u} \ge \alpha > 0$, $f(x,u) \in C^2[0,L]$, the solution of (1)-(2) satisfies the following estimates:

$$\|\,R\|_{\infty,\bar{\omega}_N} \le CN^{-1} \tag{30}$$

$$|r| \le CN^{-1} \tag{31}$$

Proof. First, we consider error function R_i,

$$|R_i| \le h_i^{-1} \int_{x_{i-1}}^{x_i} (x_i - x_{i-1}) |f'(x,u)| dx$$
$$= h_i^{-1} \int_{x_{i-1}}^{x_i} (x_i - x_{i-1}) \left| \frac{\partial f}{\partial u}\frac{dx}{dx} + \frac{\partial f\big(x,u(x)\big)}{\partial u}\frac{du}{dx} \right| dx$$
$$\le Ch_i^{-1} \int_{x_{i-1}}^{x_i} (x_i - x_{i-1}) dx + Ch_i^{-1} \int_{x_{i-1}}^{x_i} (x_i - x_{i-1}) |u'(x)| dx$$

$$\leq Ch_i + Ch_i^{-1}\int_{x_{i-1}}^{x_i}(x_i - x_{i-1})\left|1+\frac{1}{\varepsilon}e^{\frac{-\alpha x}{\varepsilon}}\right|dx$$

$$\leq C\left\{h_i + h_i^{-1}\varepsilon^{-1}\int_{x_{i-1}}^{x_i}(x_i - x_{i-1})e^{\frac{-\alpha x}{\varepsilon}}dx\right\}, 1 \leq i \leq N. \qquad (32)$$

In the first case, $x_i \in [0, \sigma]$,

$a)\ \sigma < \frac{1}{2},\ \sigma = -\alpha^{-1}\varepsilon ln\varepsilon$:

$$h_i = x_i - x_{i-1} = \alpha^{-1}\varepsilon\left[\ln\left[1-(1-\varepsilon)\frac{2i}{N}\right] - \ln\left[1-(1-\varepsilon)\frac{2(i-1)}{N}\right]\right]$$

where, according to i, we use intermediate value theorem as follows:

$$h_i = \alpha^{-1}\varepsilon\frac{2(1-\varepsilon)N^{-1}}{1-i_1 2(1-\varepsilon)N^{-1}} \leq 2\alpha^{-1}(1-\varepsilon)N^{-1} \leq CN^{-1} \qquad (33)$$

Thus, we obtain

$$\begin{aligned}|R_i| &\leq C\left\{h_i + h_i^{-1}\varepsilon^{-1}\int_{x_{i-1}}^{x_i}(x_i - x_{i-1})e^{\frac{-\alpha x}{\varepsilon}}dx\right\} \leq C\left\{h_i + h_i^{-1}\varepsilon^{-1}h_i\int_{x_{i-1}}^{x_i}e^{\frac{-\alpha x}{\varepsilon}}dx\right\}\\ &\leq C\left\{h_i + \varepsilon^{-1}\int_{x_{i-1}}^{x_i}e^{\frac{-\alpha x}{\varepsilon}}dx\right\} = C\left\{h_i + \alpha^{-1}\left[e^{\frac{-\alpha x_i}{\varepsilon}} - e^{\frac{-\alpha x_{i-1}}{\varepsilon}}\right]\right\}\\ &= C\left\{h_i + \alpha^{-1}e^{\frac{-\alpha}{\varepsilon}}\left[e^{\alpha^{-1}\varepsilon\ln\left[1-(1-\varepsilon)\frac{2i}{N}\right]} - e^{\alpha^{-1}\varepsilon\ln\left[1-(1-\varepsilon)\frac{2(i-1)}{N}\right]}\right]\right\} = C\{h_i + 2\alpha^{-1}(1-\varepsilon)N^{-1}\}\\ &\leq C\{CN^{-1} + 2\alpha^{-1}(1-\varepsilon)N^{-1}\} \leq CN^{-1}.\end{aligned}$$

$b)\ \sigma = \frac{1}{2}, \sigma = \frac{1}{2} < -\alpha^{-1}\varepsilon ln\varepsilon$:

$$h_i = x_i - x_{i-1} = \alpha^{-1}\varepsilon\left[\ln\left[1-\left(1-e^{\frac{-\alpha}{2\varepsilon}}\right)\frac{2i}{N}\right] - \ln\left[1-\left(1-e^{\frac{-\alpha}{2\varepsilon}}\right)\frac{2(i-1)}{N}\right]\right]$$

where, according to i, we use intermediate value theorem as follows:

$$h_i = \alpha^{-1}\varepsilon\frac{2(1-e^{\frac{-\alpha}{2\varepsilon}})N^{-1}}{1-i_1 2(1-e^{\frac{-\alpha}{2\varepsilon}})N^{-1}} \leq 2\alpha^{-1}N^{-1} \leq CN^{-1}. \qquad (34)$$

Hence

$$\begin{aligned}|R_i| &\leq C\left\{h_i + \alpha^{-1}\left[e^{\frac{-\alpha x_i}{\varepsilon}} - e^{\frac{-\alpha x_{i-1}}{\varepsilon}}\right]\right\} = C\left\{h_i + \alpha^{-1}e^{\frac{-\alpha}{\varepsilon}}\left[e^{\alpha^{-1}\varepsilon\ln\left[1-\left(1-e^{\frac{-\alpha}{2\varepsilon}}\right)\frac{2i}{N}\right]} - e^{\alpha^{-1}\varepsilon\ln\left[1-\left(1-e^{\frac{-\alpha}{2\varepsilon}}\right)\frac{2(i-1)}{N}\right]}\right]\right\}\\ &\leq CN^{-1}\end{aligned}$$

In the second case, $x_i \in [\sigma, 1]$, we obtain

$$|R_i| \leq C\left\{h_i + \varepsilon^{-1}\int_{x_{i-1}}^{x_i}e^{\frac{-\alpha x}{\varepsilon}}dx\right\} \leq C\left\{\frac{1}{N} + \varepsilon^{-1}h_i\right\} \leq C\{N^{-1} + \varepsilon^{-1}h^{-1}\} \leq C\{N^{-1} + \varepsilon^{-1}N^{-1}\} \leq CN^{-1}.$$

Consquently, we have

$$\| R\|_{\infty,\bar{\omega}_N} \leq CN^{-1}. \qquad (35)$$

Now, we evaluate error function$r$ as follows:

$x - x_{i-1} \leq h_i$, $b^* = max|b(x)| \leq C$, $b'(x) \leq C$ ve $u'(x) \leq C_0$

under the above assumptions,

$$|r| \leq \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} (x - x_{i-1}) \left[b'(x)u(x) + b(x)u'(x)\right] dx$$

$$\leq \sum_{i=1}^{N} h_i \int_{x_{i-1}}^{x_i} \left[C + C\left(1 + \frac{1}{\varepsilon} e^{-\frac{\alpha x}{\varepsilon}}\right)\right] dx$$

$$\leq C \sum_{i=1}^{N} h_i \int_{x_{i-1}}^{x_i} \left[1 + \frac{1}{\varepsilon} e^{-\frac{\alpha x}{\varepsilon}}\right] dx$$

$$\leq C \left\{ \sum_{i=1}^{\frac{N}{2}} h_i \int_{x_{i-1}}^{x_i} \left[1 + \frac{1}{\varepsilon} e^{-\frac{\alpha x}{\varepsilon}}\right] dx + \sum_{i=\frac{N}{2}+1}^{N} h_i \int_{x_{i-1}}^{x_i} \left[1 + \frac{1}{\varepsilon} e^{-\frac{\alpha x}{\varepsilon}}\right] dx \right\}$$

$$= C \left[h_i \int_0^{\sigma} \left[1 + \frac{1}{\varepsilon} e^{-\frac{\alpha x}{\varepsilon}}\right] dx + h_i \int_{\sigma}^{1} \left[1 + \frac{1}{\varepsilon} e^{-\frac{\alpha x}{\varepsilon}}\right] dx \right] \leq C\, h_i \tag{36}$$

In the first case, $x_i \in [0, \sigma]$:
$a)\ \sigma < \frac{1}{2},\ \sigma = -\alpha^{-1} \varepsilon ln\varepsilon,$

$$x_{i-1} = \alpha^{-1} \varepsilon \ln \left[1 - (1 - \varepsilon) \frac{2(i-1)}{N}\right]$$

and

$$h_i = x_i - x_{i-1} = \alpha^{-1} \varepsilon \left[\ln\left[1 - (1-\varepsilon)\frac{2i}{N}\right] - \ln\left[1 - (1-\varepsilon)\frac{2(i-1)}{N}\right]\right]$$

where, according to i, we use intermediate value theorem as follows:

$$h_i = \alpha^{-1} \varepsilon \frac{2(1-\varepsilon)N^{-1}}{1 - i_1 2(1-\varepsilon)N^{-1}} \leq 2\alpha^{-1}(1-\varepsilon)N^{-1} \leq CN^{-1} \tag{37}$$

Thus, we obtain

$$|r| \leq C[h_i + h_i] \leq CN^{-1}.$$

b) $\sigma = \dfrac{1}{2}, \sigma = \dfrac{1}{2} < -\alpha^{-1} \varepsilon ln\varepsilon,$

After a simple calculation we obtain,

$$h_i = \alpha^{-1} \varepsilon \frac{2(1 - e^{\frac{-\alpha}{2\varepsilon}})N^{-1}}{1 - i_1 2(1 - e^{\frac{-\alpha}{2\varepsilon}})N^{-1}} \leq 2\alpha^{-1} N^{-1} \leq CN^{-1}. \tag{38}$$

Hence

$$|r| \leq C[h_i + h_i] \leq CN^{-1}.$$

In the second case, $x_i \in [\sigma, 1]$:

a) $\sigma < \frac{1}{2}, -\alpha^{-1} \varepsilon ln\varepsilon < \frac{1}{2},$
we use the following equation,

$$h = h_i = h^2 = \frac{2(1-\sigma)}{N} \leq \frac{2}{N}, \qquad i = \overline{\frac{N}{2} + 1, N}$$

we obtain,

$$|r| \leq C[h_i + h_i] \leq C[h^1 + h^2] \leq C[N^{-1} + 2N^{-1}] \leq CN^{-1}$$

and

$$|r| \leq CN^{-1}. \tag{39}$$

Teorem 5.1. Under the assumptions of Lemma 5.1. and Lemma 5.2. Let $u(x)$, be the solution of (1)-(2) and y_i, be the solution of (18)-(19). Then, the following uniform error estimate satisfies

$$\|y-u\|_{\infty,\bar{\omega}_N} \leq CN^{-1}\,. \tag{40}$$

6. Algorithm and Numerical Results

We give some numerical results to solve the nonlinear problem (18)-(19) on Bakhvalov mesh using quasilinearization technique.

6.1. Algorithm

We present the following quasilinearization technique for the difference scheme (18)-(19):

$$\frac{\varepsilon}{h_i}y_{i-1}^{(n)}-\left(\frac{\varepsilon}{h_i}+\frac{\partial f(x_i,y_i)}{\partial y}\right)y_i^{(n)}=f(x_i,y_i)-y_i^{(n-1)}\left(\frac{\partial f(x_i,y_i)}{\partial y}\right), \qquad i=1,\dots,N$$

$$y_i^{(n)}=\frac{f(x_i,y_i)-y_i^{(n-1)}\left(\frac{\partial f(x_i,y_i)}{\partial y}\right)-\frac{\varepsilon}{h_i}y_{i-1}^{(n)}}{-\left(\frac{\varepsilon}{h_i}+\frac{\partial f(x_i,y_i)}{\partial y}\right)}, \qquad i=1,..,N$$

$$y_0^{(n)}=\beta y_N^{(n-1)}+\sum_{i=1}^{N}h_ib_iy_i^{(n-1)}+d, \qquad n=1,2,\dots$$

6.2. Numerical Results

We give some examples to see the effectiveness of the presented method

Example 1

$$\varepsilon u'+2u-e^{-u}+x^2=0, \;\; 0<x<1 \tag{41}$$

$$u(0)=\frac{1}{2}u(1)+\frac{1}{4}\int_0^1 e^{-s}u(s)ds+1 \tag{42}$$

this problem has not the exact solution. Thus, we use ω −mesh to calculate the errors and convergence rates, respectively.

$$e^N=\max_{\varepsilon}e_\varepsilon^N\;,\; e_\varepsilon^N=\max_i|y_i^N-y_{2i}^{2N}|, \qquad P_\varepsilon^N=\log_2\left(\frac{e^N}{e^{2N}}\right)$$

where, $y_i^N$and y_{2i}^{2N} are the numerical solutions for $N$and $2N$.

According to (6.1), we give the algorithm as follows:

$$\frac{\varepsilon}{h_i}y_{i-1}^{(n)}-\left(\frac{\varepsilon}{h_i}+2+e^{-y_i^{(n-1)}}\right)y_i^{(n)}=2y_i^{(n-1)}+x_i^2-y_i^{(n-1)}\left(2+e^{-y_i^{(n-1)}}\right)-e^{-y_i^{(n-1)}}$$

$$y_i^{(n)}=\frac{\frac{\varepsilon}{h_i}y_{i-1}^{(n)}-(2y_i^{(n-1)}+x_i^2-y_i^{(n-1)}\left(2+e^{-y_i^{(n-1)}}\right)-e^{-y_i^{(n-1)}})}{\frac{\varepsilon}{h_i}+2+e^{-y_i^{(n-1)}}}, i=1,..,N$$

$$y_0^{(n)} = \frac{1}{2} y_N^{(n-1)} + \sum_{i=1}^{N} h_i b_i y_i^{(n-1)} + 1, \qquad n = 1,2,\dots$$

the initial guess in the iteration procedure is $y_i^{(0)} = 0.5$,
The stopping criterion is taken as $\left| y_i^{(n)} - y_i^{(n-1)} \right| \le 10^{-5}$.

The numerical results obtained from the problem of the difference scheme by comparison, the error and uniform rates of convergence were found and these are shown in Table 1. Consequently, numerical results show that the proposed scheme is working very well.

Table 1. The computed maximum pointwise errors e_N and e_{2N}, the numerical rate of convergence p_N on the Bakhvalow mesh $\bar{\omega}_N$ for different values of $N$and ε.

ε/N	8	16	32	64
2^{-1}	0.0364705764 p=0.76	0.0213939341 p=0.88	0.0115893393 p=1.05	0.0051560120
2^{-2}	0.0457353908 p=0.79	0.0263063297 p=0.89	0.0141504831 p=0.94	0.0073451227
2^{-3}	0.0547795035 p=0.79	0.0316355902 p=0.87	0.0172632555 p=0.93	0.0090159047
2^{-4}	0.0592489265 p=0.74	0.0354285745 p=0.87	0.0192899598 p=0.93	0.0100776157
2^{-5}	0.0617440905 p=0.71	0.0376291853 p=0.87	0.0204860025 p=0.93	0.0107447933
2^{-6}	0.0632177961 p=0.70	0.0389122636 p=0.87	0.0212015157 p=0.92	0.0111341923
2^{-7}	0.0640959677 p=0.69	0.0396529864 p=0.87	0.0216490391 p=0.93	0.0113563269
2^{-8}	0.0646130925 p=0.68	0.0400746536 P=0.87	0.0219009225 p=1.00	0.010926500

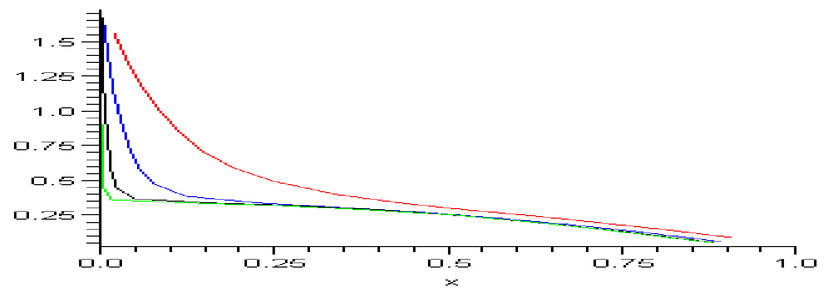

Figure 1. Exact solution distribution for $N = 16$ and $\varepsilon = 2^{-4}, 2^{-6}, 2^{-8}$

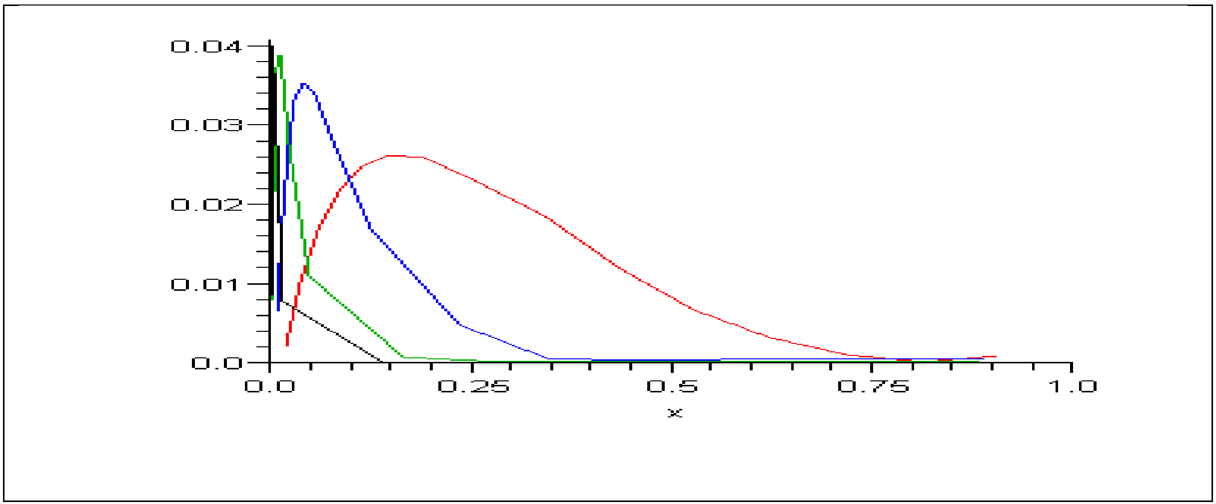

Figure 2. Error distribution for $N = 16$ and $\varepsilon = 2^{-2}, 2^{-4}, 2^{-6}, 2^{-8}$

From the graps it is show that the error is maximum near the boundary layer and it is almost zero in outer region in the Figure 2. When ε −values are small, graphic curves leaned more towards the coordinate axes in Figure 1.

Consequently, the purpose of this study was to give uniform finite difference method for numerical solution of nonlinear singularly perturbed problem with nonlocal boundary conditions. The numerical method was constructed on Bakhvalov mesh. The method was pointed out to be convergent, uniformly in the ε-parameter, of first order in the discrete maximum norm. The numerical example illustrated in practice the result of convergence proved theoretically.

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